Program:	RFEM 5, RSTAB 8, RF-DYNAM Pro, DYNAM Pro
Category:	**Geometrically Linear Analysis, Isotropic Linear Elasticity, Dynamics, Contact, Friction, Structural Nonlinearity, Member**
Verification Example:	**0116 – Vibrations with Coulomb Friction**

0116 – Vibrations with Coulomb Friction

Description

A simple oscillator consists of mass m (considered only in x-direction) and linear spring of stiffness k. The mass is embedded on a surface with Coulomb friction coefficient μ and is loaded by constant-in-time axial F_x and transversal F_z forces according to **Figure 1**. Calculate the time response of the system. The problem is described by the following set of parameters, see also [1].

System Properties		Mass	m	100.000	kg
		Spring Stiffness	k	5.000	kN/m
		Friction Coefficient	μ	0.100	–
Load	Forces	Axial Force	F_x	1.500	kN
		Transversal Force	F_z	1.000	kN

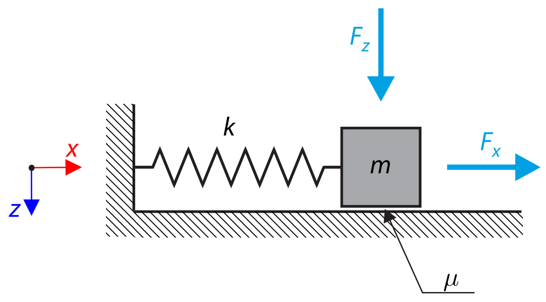

Figure 1: Problem Sketch

Determine the deflection u_x at times $t = 1$ s and $t = 2$ s, and the residual deflection $u_{x,\text{res}}$ when the spring reaches its equilibrium state.

Analytical Solution

The problem is described by the equation of motion – a second-order differential equation, which is non-linear due to the sign function, that ensures the correct orientation of the frictional force depending on the velocity $\dot{u}_x$, namely

$$m\ddot{u}_x + \mu F_z \text{sgn}(\dot{u}_x) + k u_x = F_x \qquad (116 – 1)$$

The solution of **(116 – 1)** can be separated into time intervals $[t_{i-1}, t_i], i \in \mathbb{N}, t_0 = 0$, where the velocity $\dot{u}_x$ does not change sign (i.e., movement either to the right, or to the left), namely

$$u_x(t) := u_x^{(i)}(t), \text{ if } t \in [t_{i-1}, t_i], \tag{116 – 2}$$

where $t_i > 0, i \in \mathbb{N}$, are such that $\dot{u}_x(t_i) = 0$, and $u_x^{(i)}$ is the solution of the second–order ordinary differential equation in (t_{i-1}, t_i)

$$\ddot{u}_x^{(i)} + \Omega^2 u_x^{(i)} = \begin{cases} \frac{1}{m}(F_x - \mu F_z) =: p_-, \text{ if } i \text{ odd}, \\ (F_x + \mu F_z) =: p_+, \text{ if } i \text{ even}, \end{cases} \tag{116 – 3}$$

$$u_x^{(i)}(t_{i-1}) = u_x^{(i-1)}(t_{i-1}), \tag{116 – 4}$$

$$\dot{u}_x^{(i)}(t_{i-1}) = 0. \tag{116 – 5}$$

Note that $\Omega = \sqrt{k/m}$ is the angular frequency and let $u_x^{(0)} \equiv 0$. In the first part, the movement is in the positive x-axis direction, the following solution is obtained

$$u_x^{(1)} = -\frac{p_-}{\Omega^2}\cos(\Omega t) + \frac{p_-}{\Omega^2}, \tag{116 – 6}$$

$$t_1 = \frac{\pi}{\Omega}. \tag{116 – 7}$$

For the second step the solution is

$$u_x^{(2)} = \left(\frac{2p_-}{\Omega^2} - \frac{p_+}{\Omega^2}\right)\cos(\Omega(t - t_1)) + \frac{p_+}{\Omega^2} \tag{116 – 8}$$

Further calculations are carried out analogously. The equation **(116 – 3)** is alternately used until the end of the movement. The movement stops when the spring force is less than the tangential and axial force. Two cases, for the movement to the right (i is odd) and to the left (i is even), are possible. The phase, when the movement stops, can be determined as follows

$$i \leq \min\left\{\frac{\Omega^2}{p_+ - p_-}\left(x_s + \frac{\mu F_z}{k}\right), \frac{\Omega^2}{p_- - p_+}\left(x_s - \frac{\mu F_z}{k} - \frac{p_- + p_+}{\Omega^2}\right)\right\}, \tag{116 – 9}$$

where x_s is the statical displacement caused by the force F_x,

$$x_s = \frac{F_x - \mu F_z}{k} \approx 280.000 \text{ m}. \tag{116 – 10}$$

In this case, the result of the relation **(116 – 9)** is $i = 7$. Please note that $i \in \mathbb{N}$. The movement stops at the end of the seventh phase. Hence, the residual deflection of the mass $u_{x,\text{res}}$ is

$$u_{x,\text{res}} = u_x^{(7)}(4) = \left(-\frac{7p_1}{\Omega^2} + \frac{6p_2}{\Omega^2}\right)\cos(\Omega(7t_1 - 6t_1)) + \frac{p_1}{\Omega^2} \approx 320.000 \text{ mm}. \tag{116 – 11}$$

Furthermore, the deflections in given test time are calculated.

$$u_x(1) = \left(-\frac{3p_-}{\Omega^2} + \frac{2p_+}{\Omega^2}\right)\cos(\Omega(1-2t_1)) + \frac{p_-}{\Omega^2} \approx 138.930 \text{ mm} \qquad (116-12)$$

$$u_x(2) = \left(-\frac{5p_-}{\Omega^2} + \frac{4p_+}{\Omega^2}\right)\cos(\Omega(2-4t_1)) + \frac{p_-}{\Omega^2} \approx 280.596 \text{ mm} \qquad (116-13)$$

RFEM 5 and RSTAB 8 Settings

- Modeled in RFEM 5.10.00 and RSTAB 8.10.00
- The element size is $l_{FE} = 0.010$ m
- The number of increments is 10
- Isotropic linear elastic model is used

The comparison of the analytical solution and RFEM 5 / RSTAB 8 solution can be seen in **Figure 2**.

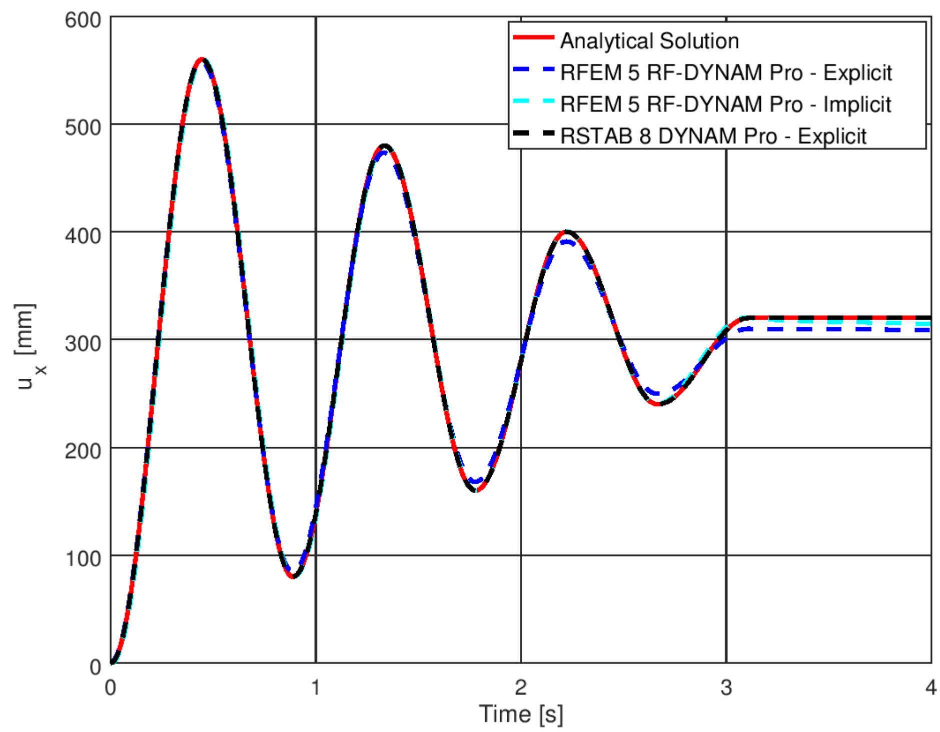

Figure 2: Analytical and RFEM 5 / RSTAB 8 solution

Results

Structure Files	Program	Solution Method
0116.01	RFEM 5 – RF-DYNAM Pro	Explicit analysis
0116.02	RFEM 5 – RF-DYNAM Pro	Nonlinear implicit Newmark analysis
0116.03	RSTAB 8 – DYNAM Pro	Explicit analysis

Model	Analytical Solution	RFEM 5 / RSTAB 8	
	$u_x(1)$ [mm]	$u_x(1)$ [mm]	Ratio [-]
RFEM 5, Explicit analysis	138.930	143.780	1.035
RFEM 5, Nonlinear implicit Newmark analysis		134.923	0.971
RSTAB 8, Explicit analysis		138.642	1.000

Model	Analytical Solution	RFEM 5 / RSTAB 8	
	$u_x(2)$ [mm]	$u_x(2)$ [mm]	Ratio [-]
RFEM 5, Explicit analysis	280.596	281.484	1.003
RFEM 5, Nonlinear implicit Newmark analysis		279.921	0.998
RSTAB 8, Explicit analysis		280.097	0.998

Model	Analytical Solution	RFEM 5 / RSTAB 8	
	$u_{x,\mathrm{res}}(4)$ [mm]	$u_{x,\mathrm{res}}(4)$ [mm]	Ratio [-]
RFEM 5, Explicit analysis	320.000	309.011	0.966
RFEM 5, Nonlinear implicit Newmark analysis		314.470	0.983
RSTAB 8, Explicit analysis		320.000	1.000

References

[1] STEJSKAL, V. and OKROUHLÍK, M. *Kmitání s Matlabem*. Vydavatelství ČVUT Praha.